

**$\beta^4$  potential at the U(5)–O(6) critical point of the interacting boson model**José Enrique García-Ramos,<sup>1,\*</sup> Jorge Dukelsky,<sup>2,†</sup> and José M. Arias<sup>3,‡</sup><sup>1</sup>*Departamento de Física Aplicada, Universidad de Huelva, E-21071 Huelva, Spain*<sup>2</sup>*Instituto de Estructura de la Materia, CSIC, Serrano 123, E-28006 Madrid, Spain*<sup>3</sup>*Departamento de Física Atómica, Molecular y Nuclear, Facultad de Física, Universidad de Sevilla, Apartado 1065, E-41080 Sevilla, Spain*

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Exact numerical results of the interacting boson model Hamiltonian along the integrable line from U(5) to O(6) are obtained by diagonalization within boson seniority subspaces. The matrix Hamiltonian reduces to a block tridiagonal form that can be diagonalized for large number of bosons. We present results for the low-energy spectrum and the transition probabilities for systems up to 10,000 bosons, which confirm that at the critical point the system is equally well described by the Bohr Hamiltonian with a  $\beta^4$  potential.

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The main goal of this Brief Report is to report on new results that complete a previous study [1] on the relations between the critical point in the transition from U(5) to O(6) limits of the interacting boson model (IBM) [2] and the recently proposed E(5) critical point symmetry [3]. In Ref. [1] two different boson Hamiltonians performing the transition from U(5) to O(6) were used to show that at the critical point (i) they provide different spectra and transitions for small number of bosons (ii) they converge to the same spectrum for large  $N$ , and (iii) both converge to the spectrum provided by the Bohr Hamiltonian [4] with a  $\beta^4$  potential rather than to the one provided by a square-well potential as in the  $E(5)$  model. Most of the large- $N$  analysis was based on the solution of the Richardson equations [5,6], which allow us to obtain energy eigenvalues, but the form of the eigenstates is not well suited to calculate transition probabilities. Therefore, to study transition rates we had to resort to current IBM codes that restricted our calculations to systems up to  $N = 40$  [1]. For these small- $N$  values, the transition rates show a tendency to approach the  $\beta^4$  potential results but they are not conclusive. In this Brief Report we present an alternative to the Richardson equations for obtaining the low-energy eigenvalues and, on the same footing, the transition probabilities in the U(5)–O(6) transitional region for large  $N$  values.

The two Hamiltonians describing the U(5)–O(6) transition studied in Ref. [1] are

$$\hat{H}_I = x\hat{n}_d + \frac{1-x}{N-1}\hat{P}^\dagger\hat{P} \quad (1)$$

and

$$\hat{H}_{II} = x\hat{n}_d - \frac{1-x}{N}\hat{Q}^{\chi=0} \cdot \hat{Q}^{\chi=0}, \quad (2)$$

where

$$\hat{n}_d = \sum_m d_m^\dagger d_m, \quad (3)$$

$$\hat{P}^\dagger = \frac{1}{2}(d^\dagger \cdot d^\dagger - s^\dagger \cdot s^\dagger) = \frac{1}{2}(P_d^\dagger - P_s^\dagger), \quad (4)$$

$$\hat{Q}^{\chi=0} = (s^\dagger \times \tilde{d} + d^\dagger \times \tilde{s})^{(2)}, \quad (5)$$

and  $\cdot$  stands for the scalar product. We have introduced in (4) the boson-pair creation operators  $P_d^\dagger = d^\dagger \cdot d^\dagger$  and  $P_s^\dagger = s^\dagger \cdot s^\dagger$  that will be used later on.

The mean-field analysis of the quantum phase diagram of the IBM is usually performed within the intrinsic state formalism [7,8] where, after separating the three Euler angles, the trial wave function is a boson condensate depending on the two geometrical variables  $\beta$  and  $\gamma$ . Along the U(5)–O(6) transition the energy surface is  $\gamma$  independent and the intrinsic ground-state energy for a given value of the control parameter  $x$  corresponds to the value of the deformation parameter  $\beta$ , which minimizes the energy surface. The phase transition along this line is then signaled by the condition

$$[d^2 E(N, \beta)/d\beta^2]_{\beta=0} = 0, \quad (6)$$

which fixes the critical value of the control parameter  $x$ . For the Hamiltonian (1) the critical  $x$  is  $x_c^I = 0.5$ , independent of the number of bosons  $N$ , whereas for the Hamiltonian (2) it is  $x_c^{II} = (4N - 8)/(5N - 8)$ . In the large  $N$  limit  $x_c^{II} \rightarrow 4/5$ .

To study the eigenstates of the Hamiltonians (1) and (2) we introduce the  $s$ - and  $d$ -boson pair algebra [9]

$$K_s^+ = \frac{1}{2}s^\dagger \cdot s^\dagger = \frac{1}{2}P_s^\dagger = (K_s^-)^\dagger, \quad (7)$$

$$K_s^0 = \frac{1}{2}\left(s^\dagger s + \frac{1}{2}\right) = \frac{1}{2}\hat{n}_s + \frac{1}{4},$$

$$K_d^+ = \frac{1}{2}d^\dagger \cdot d^\dagger = \frac{1}{2}P_d^\dagger = (K_d^-)^\dagger, \quad (8)$$

$$K_d^0 = \frac{1}{2}\sum_m \left(d_m^\dagger d_m + \frac{1}{2}\right) = \frac{1}{2}\hat{n}_d + \frac{5}{4}.$$

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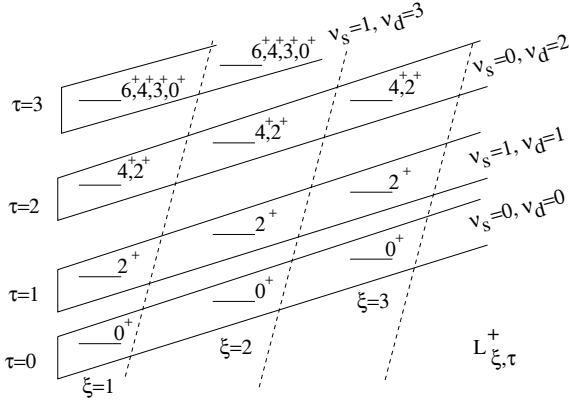


FIG. 1. Schematic spectrum obtained by diagonalization within boson seniority subspaces as explained in the text and its correspondence with the one of Refs. [3] and [1].

For each  $\ell$  value, 0 or 2, the three operators  $\{K_\ell^+, K_\ell^-, K_\ell^0\}$  satisfy the  $su(1, 1)$  commutator algebra

$$[K_\ell^0, K_\ell^\pm] = \pm \delta_{\ell\ell'} K_\ell^\pm, \quad [K_\ell^+, K_\ell^-] = -2\delta_{\ell\ell'} K_\ell^0. \quad (9)$$

A complete set of eigenstates for a general IBM U(5)–O(6) transitional Hamiltonian can be written in terms of the raising operator  $K_\ell^+$  acting on a subspace of unpaired bosons characterized by the seniority quantum number  $\nu$ ,

$$|\tilde{n}_\ell \nu_\ell\rangle = \frac{1}{\sqrt{C_{\ell, \nu_\ell}^{\tilde{n}_\ell}}} (K_\ell^+)^{\tilde{n}_\ell} |\nu_\ell\rangle, \quad (10)$$

where  $\nu_s = 0, 1$ ;  $\nu_d = 0, 1, 2, \dots$ ; and  $|\nu_\ell\rangle$  is a normalized state. The value of  $\nu_\ell$  gives the number of bosons of type  $\ell$  not coupled in pairs to zero angular momentum. The label  $\tilde{n}_\ell$  refers to boson pairs coupled to zero angular momentum. Therefore, the total number of bosons is  $N = 2\tilde{n}_s + 2\tilde{n}_d + \nu_s + \nu_d$ . Using the  $su(1, 1)$  algebra it is straightforward to

obtain the normalization constants

$$C_{\ell, \nu_\ell}^{\tilde{n}_\ell} = \langle \nu_\ell | (K_\ell^-)^{\tilde{n}_\ell} (K_\ell^+)^{\tilde{n}_\ell} | \nu_\ell \rangle = \frac{\tilde{n}_\ell! (2\tilde{n}_\ell + 2\ell + 2\nu_\ell - 1)!!}{2^{\tilde{n}_\ell} (2\ell + 2\nu_\ell - 1)!!}.$$

Now we proceed to construct the complete set of states as

$$|\tilde{n}_s \tilde{n}_d, \nu_s \nu_d\rangle = \frac{1}{\sqrt{C_{s, \nu_s}^{\tilde{n}_s} C_{d, \nu_d}^{\tilde{n}_d}}} (K_s^+)^{\tilde{n}_s} (K_d^+)^{\tilde{n}_d} |\nu_s \nu_d\rangle. \quad (11)$$

The basis (11), although lacking information on angular momentum, is especially useful for diagonalizing the Hamiltonians (1) and (2). To show this, we rewrite the Hamiltonians (1) and (2) in terms of the generators of the two  $su(1, 1)$  algebras,

$$\hat{H}_I = x\hat{n}_d + \frac{1-x}{(N-1)} (K_s^+ K_s^- + K_d^+ K_d^- - K_s^+ K_d^- - K_d^+ K_s^-), \quad (12)$$

$$\hat{H}_{II} = x\hat{n}_d - \frac{1-x}{N} (4K_s^+ K_d^- + 4K_d^+ K_s^- + 5\hat{n}_s + \hat{n}_d + 2\hat{n}_s \hat{n}_d). \quad (13)$$

The matrix elements of the relevant operators for both Hamiltonians in the basis (11) are

$$\begin{aligned} \langle \tilde{n}_s \tilde{n}_d, \nu_s \nu_d | \hat{n}_s | \tilde{n}_s \tilde{n}_d, \nu_s \nu_d \rangle &= 2\tilde{n}_s + \nu_s, \\ \langle \tilde{n}_s \tilde{n}_d, \nu_s \nu_d | \hat{n}_d | \tilde{n}_s \tilde{n}_d, \nu_s \nu_d \rangle &= 2\tilde{n}_d + \nu_d, \\ \langle \tilde{n}_s \tilde{n}_d, \nu_s \nu_d | K_s^+ K_s^- | \tilde{n}_s \tilde{n}_d, \nu_s \nu_d \rangle &= \tilde{n}_s (\tilde{n}_s + \nu_s - \frac{1}{2}), \\ \langle \tilde{n}_s \tilde{n}_d, \nu_s \nu_d | K_d^+ K_d^- | \tilde{n}_s \tilde{n}_d, \nu_s \nu_d \rangle &= \tilde{n}_d (\tilde{n}_d + \nu_d + \frac{3}{2}), \\ \langle (\tilde{n}_s - 1)(\tilde{n}_d + 1), \nu_s \nu_d | K_d^+ K_s^- | \tilde{n}_s \tilde{n}_d, \nu_s \nu_d \rangle &= \frac{1}{2} \sqrt{\tilde{n}_s (\tilde{n}_d + 1) (2\tilde{n}_s + 2\nu_s - 1) (2\tilde{n}_d + 2\nu_d + 5)}. \end{aligned} \quad (14)$$

The Hamiltonians (1) and (2) do not mix states with different seniority quantum numbers ( $\nu_s, \nu_d$ ), leaving invariant these seniority subspaces. Within each subspace the Hamiltonian matrices are tridiagonal and can be easily diagonalized for very large  $N$  values. We will label states within each subspace

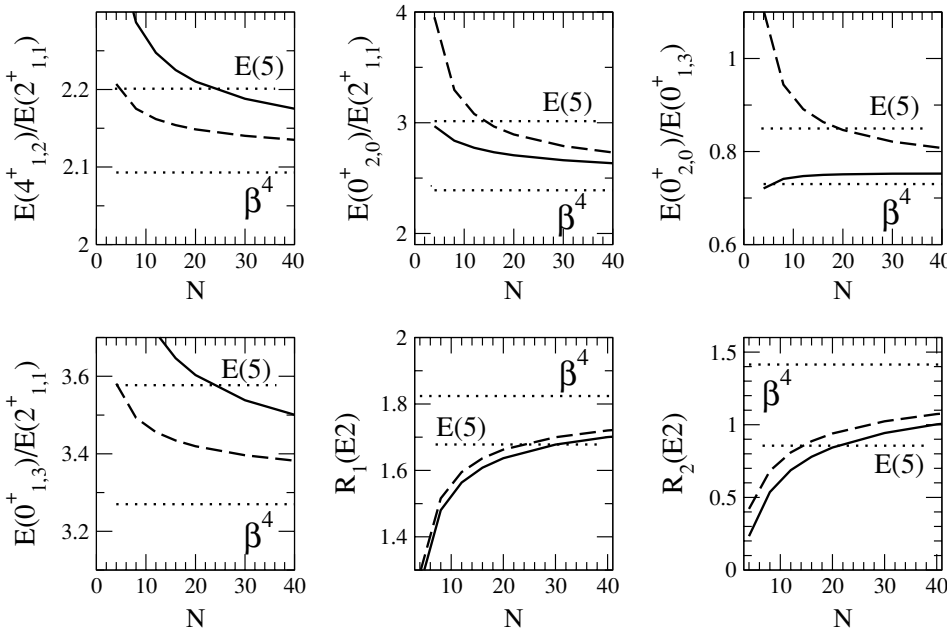


FIG. 2. Variation with the number of bosons (up to  $N = 40$ ) of selected energy and  $B(E2)$  ratios for IBM calculations performed at the critical points of Hamiltonians (1) (broken line) and (2) (full line). The corresponding  $E(5)$  and  $\beta^4$  values are marked with horizontal dotted lines.

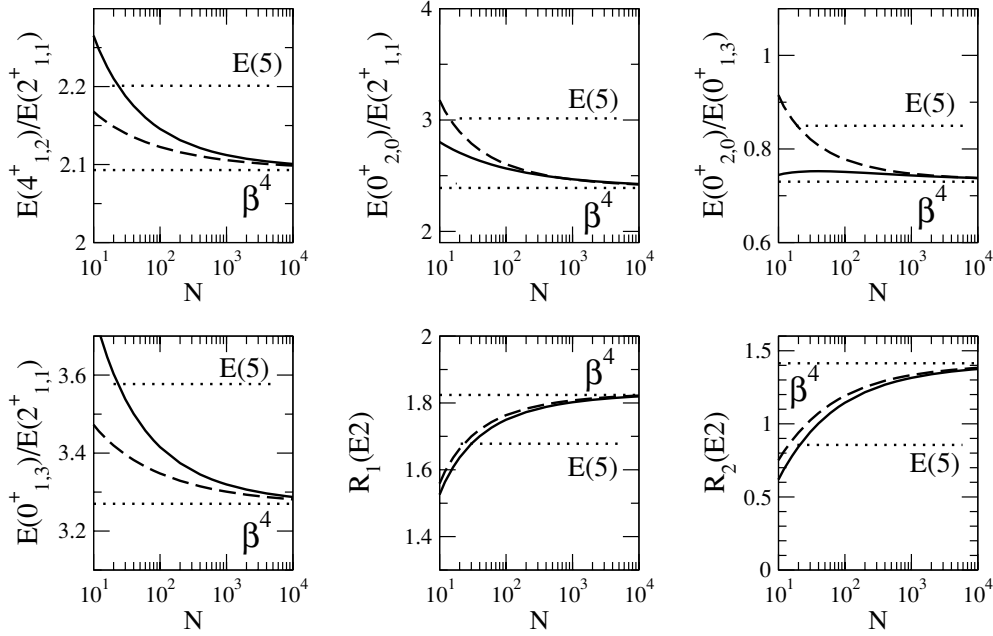


FIG. 3. Same as Fig. 2 but here the number of bosons runs up to 10,000 in both the energy and the  $B(E2)$  ratios. Note the logarithmic scale for the  $N$  axis.

by the quantum number  $\xi$ . It is worthwhile to note here that  $d$ -boson seniority,  $\nu_d$ , is equivalent to the O(5) quantum number  $\tau$  [2]. The construction of the spectrum for a system with even number of bosons is as follows: One starts with the subspace  $\tau = 0$  ( $\nu_s = 0, \nu_d = 0$ ), where all the bosons are coupled in pairs of zero angular momentum. Consequently, states within this subspace will have total angular momentum  $L = 0$ . The lowest eigenvalue ( $\xi = 1$ ) is the ground state  $0_{1,0}^+$  (the notation is  $L_{\xi,\tau}^\pi$  [3]), the second lowest eigenvalue is the first excited state  $\tau = 0, L^\pi = 0^+$ , which is labeled  $0_{2,0}^+$ , etc. The next block with  $\tau = 1$  ( $\nu_s = 1, \nu_d = 1$ ) has one pair broken into an  $s$  boson and a  $d$  boson. Correspondingly, all states in this block have  $L = 2$ . The lowest eigenvalue ( $\xi = 1$ ) is the lowest  $2^+$ , which is labeled as  $2_{1,1}^+$ , the next one ( $\xi = 2$ ) is  $2_{2,1}^+$ , etc. The next block is for  $\tau = 2$  ( $\nu_s = 0, \nu_d = 2$ ) and corresponds to one boson pair broken into two  $d$  bosons. It provides states with angular momenta  $L = 4, 2$ . (Notice that  $L = 0$  is excluded from this subspace since it is included in the  $\nu_s = 0, \nu_d = 0$  subspace.) One can continue in this way with the next block,  $\tau = 3$  ( $\nu_s = 1, \nu_d = 3$ ), which corresponds to two boson pairs broken into one  $s$  boson and three  $d$  bosons and gives rise to  $L = 6, 4, 3, 0$  states and so on. The ground-state band is formed by all lowest ( $\xi = 1$ ) eigenstates for  $\tau = 0, 1, 2, 3, \dots$ . The first excited band ( $\xi = 2$ ) is formed by the next lowest  $\tau = 0, 1, 2, 3, \dots$  eigenstates, etc. Following this sequence one finds the well-known triangular structure associated to O(5). All this is shown schematically in Fig. 1.

The diagonalization of the Hamiltonian in each subspace provides the necessary information to calculate electromagnetic transition rates. We will be interested here in the electric quadrupole transitions, which, apart from an unimportant scale factor, are described by the quadrupole operator (5). The action

of this operator on the basis states without broken pairs is

$$\hat{Q}_\mu |\tilde{n}_s \tilde{n}_d, 0, 0\rangle = \frac{1}{\sqrt{C_{s,0}^{\tilde{n}_s} C_{d,0}^{\tilde{n}_d}}} \left[ \tilde{n}_s (K_s^+)^{\tilde{n}_s-1} (K_d^+)^{\tilde{n}_d} s^\dagger d_\mu^\dagger |0\rangle + \tilde{n}_d (K_s^+)^{\tilde{n}_s} (K_d^+)^{\tilde{n}_d-1} s^\dagger d_\mu^\dagger |0\rangle \right], \quad (15)$$

where  $|0\rangle$  stands for the boson vacuum.

The matrix elements of interest if one wants to evaluate transition rates from the ground state to the first excited state are

$$\langle (\tilde{n}_s - 1) \tilde{n}_d, 1, 1 | \hat{Q}_\mu | \tilde{n}_s \tilde{n}_d, 0, 0 \rangle = \sqrt{\frac{2\tilde{n}_s(2\tilde{n}_d + 5)}{5}}, \quad (16)$$

$$\langle \tilde{n}_s (\tilde{n}_d - 1), 1, 1 | \hat{Q}_\mu | \tilde{n}_s \tilde{n}_d, 0, 0 \rangle = \sqrt{\frac{2\tilde{n}_d(2\tilde{n}_s + 1)}{5}}. \quad (17)$$

If we write the eigenstates as

$$|\Psi, \nu_s \nu_d\rangle = \sum_{\tilde{n}_s, \tilde{n}_d} \zeta_{\tilde{n}_s, \tilde{n}_d}^{\nu_s \nu_d} |\tilde{n}_s \tilde{n}_d, \nu_s \nu_d\rangle, \quad (18)$$

the matrix element of the  $\hat{Q}$  operator between the ground state,  $|\Psi, 00\rangle$ , and the first excited state,  $|\Phi, 11\rangle$ , is

$$\langle \Phi, 11 | \hat{Q}_\mu | \Psi, 00 \rangle = \sum_{\tilde{n}_s, \tilde{n}_d} \left[ \sqrt{\frac{2\tilde{n}_s(2\tilde{n}_d + 5)}{5}} \zeta_{\tilde{n}_s, \tilde{n}_d}^{00} \zeta_{\tilde{n}_s-1, \tilde{n}_d}^{11} + \sqrt{\frac{2\tilde{n}_d(2\tilde{n}_s + 1)}{5}} \zeta_{\tilde{n}_s, \tilde{n}_d}^{00} \zeta_{\tilde{n}_s, \tilde{n}_d-1}^{11} \right]. \quad (19)$$

The matrix elements of the electric quadrupole operator between the first excited state ( $\nu_s = 1, \nu_d = 1$ ) and the states with  $\nu_s = 0, \nu_d = 2$  can be calculated in a similar way.

In Fig. 2 we present some selected low-energy eigenvalues and  $B(E2)$  ratios for boson numbers up to  $N = 40$  at the critical points of both IBM Hamiltonians (1) and (2). We emphasize here that the critical points for the two Hamiltonians are different. The four energy ratios presented are written explicitly in the figure and the two displayed  $B(E2)$  ratios are  $R_1 = B(E2; 4_{1,2}^+ \rightarrow 2_{1,1}^+)/B(E2; 2_{1,1}^+ \rightarrow 0_{1,0}^+)$  and  $R_2 = B(E2; 0_{2,0}^+ \rightarrow 2_{1,1}^+)/B(E2; 2_{1,1}^+ \rightarrow 0_{1,0}^+)$ , where we are using the notation  $L_{\xi,\tau}^\pi$  to indicate the states. The purpose of this figure is to correct a mistake we had in Fig. 3 of Ref. [1], where the results for the Hamiltonian (2) were calculated with a wrong value for  $x_c$ . As can be seen in the figure here, there are sizable differences in the spectrum and transition rates between both Hamiltonians at the critical points. Though from Fig. 2 a general tendency for convergence to the solution of the Bohr equation with a  $\beta^4$  potential rather than to the  $E(5)$  symmetry is observed, the results, especially from the  $B(E2)$ 's, are not yet conclusive. In Fig. 3 we show the new results of this report. The same quantities as in Fig. 2 are plotted for  $N$  values up to 10,000, including transition rates. In Ref. [1] energy eigenvalues were calculated up to  $N = 1000$  and transition

rates up to  $N = 40$ . We can now clearly appreciate, both from the energies and  $B(E2)$  transitions, that the IBM Hamiltonians at the U(5) to O(6) critical point in the large- $N$  limit converge to the Bohr Hamiltonian with a  $\beta^4$  potential.

In this Brief Report we make use of the property that the IBM Hamiltonian along the transitional line from U(5) to O(6) is block diagonal with respect to the boson seniority quantum number and tridiagonal within each subspace. This reduction allows us to obtain exact solutions up to very large number of bosons for energies and wave functions. We have applied this formalism to confirm previous studies about the correspondence between the IBM Hamiltonians at the critical point in the U(5)–O(6) transition and the Bohr Hamiltonian with a  $\beta^4$  potential for the low-energy properties. This issue has also been studied recently from a different point of view by Rowe *et al.* [10]. The formalism presented here for the IBM can be easily generalized to other two-level boson models [11].

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